Problems from the Sixteenth British Combinatorial Conference

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This document contains problems presented the 16th British Combinatorial Conference, including those submitted in advance and those handed to the editor during the conference. Please send comments on individual problems to their proposers, with a copy to the editor.

The BCC numbers of the problems will not change, though problems which are solved or withdrawn may be removed from the list in future revisions.

Problem 1 (BCC16.1): Hamiltonian planar cubic graphs. Posed by S. Jendrol' and Z. Skupień.
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Let G be a cubic bipartite 3-connected planar graph whose edge set $E(G)$ can be partitioned into three subsets $E(G) = E_1 \cup E_2 \cup E_3$ such that $\langle E_1 \rangle$ and $\langle E_2 \rangle$ are trees and $\langle E_3 \rangle$ is a cycle. Prove that G is hamiltonian.

Remark. This is true if E_2 is empty (Halin) or if E_2 consists of a single edge (Skupień).

Problem 2 (BCC16.2): Quasi Hamiltonian-Connected Graphs. Posed by M. Alabdullatif.
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Consider a Hamiltonian graph with the following property: for each pair $\{u, v\}$ of non-adjacent vertices, there exists a hamiltonian path joining u and v. Call such graph *quasi hamiltonian-connected* (*QHc*).

Unlike a hamiltonian-connected graph, a QHc graph may have connectivity 2. It has been shown that a k-regular QHc graph is necessarily 3-connected for $k = 3$ or 4 (see [1]).

Problem. Let G be a k-regular QHc graph ($k = 3$ or 4). Is G necessarily hamiltonian-connected?

The proposer checked the case when G is of order less than or equal to 10, and it turned out that G is hamiltonian-connected.

Remark. Counterexample to this problem, both for $k = 3$ and for $k = 4$, have been found by Gunnar Brinkmann. The smallest counterexample has 16 vertices. (This information is from Stefan Brandt.)

Editor's Note. In the preliminary version, the proposer asked whether a 3-connected k-regular QHc graph is necessarily hamiltonian-connected? A negative answer was given by Stefan Brandt at the conference. He constructed, for each k, a k-connected, $(2k-1)$-regular graph which is QHc but not hamiltonian-connected.

[1] M. Alabdullatif, On Quasi-Hamiltonian-Connected Graphs, *JCISS*, to appear.

Problem 3 (BCC16.3): A condition for pancyclicity. Posed by Uwe Schelten and Ingo Schiermeyer.

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The *k-closure* $C_k(G)$ of a graph G was defined by Bondy and Chvátal (1976): recursively join all pairs of non-adjacent vertices whose degrees have sum at least k. They showed that, for an n-vertex graph G,

If $C_n(G) = K_n$ *then* G *is Hamiltonian.*

Faudree, Flandrin, Favaron and Li (1992) showed that

If $C_{n+1}(G) = K_n$ *then* G *is pancyclic.*

The examples $G = K_{n/2,n/2}$ (and many others) show that it is not true that if $C_n(G) = K_n$ then G is pancyclic.

Conjecture. If $C_n(G) = K_n$ and n is odd then G is pancyclic.

Problem 4 (BCC16.4): Cutsets in bridged graphs. Posed by Geňa Hahn.

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A graph G is *bridged* if every cycle of length of at least 4 has a bridge, that is, if every cycle C of length 4 contains two vertices u and v such that $d_G(u, v) < d_C(u, v)$ (distance).

Conjecture. For any pair of vertices u, v in a minimum cutset in a bridged graph G, $d_G(u, v) \leq 2$.

Problem 5 (BCC16.5): 4-cycles in regular spanning subgraphs. Posed by Ian Wanless.

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Let n and k be integers satisfying $n \geq 2k$. Let G be a k-regular spanning subgraph of $K_{n,n}$ and let $s(G)$ be the number of 4-cycles in G. Suppose that G is chosen to maximise $s(G)$.

Conjecture. G contains $K_{k,k}$ as a component.

Notes. The conjecture is easily proved if
(1) k divides n, or
(2) n is sufficiently large.
If it is true *in toto*, then G is determined up to isomorphism. This follows from the fact that if G maximises $s(G)$ then $s(\overline{G})$ is also maximised, where $\overline{G}$ denotes the bipartite complement of G.

A weaker result along the same lines would be to show that G is necessarily disconnected.

Problem 6 (BCC16.6): Alternating cycles in 2-arc-coloured tournaments. Posed by G. Gutin, B. Sudakov and A. Yeo.

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A cycle C in a 2-arc-coloured digraph is alternating if any two consecutive arcs in C have different colours.

Problem. Does there exist a polynomial algorithm to check whether a 2-arc-coloured tournament has an alternating cycle?

The same problem for 2-arc-coloured digraphs is proved to be NP-complete.

Problem 7 (BCC16.7): Monochromatically absorbing sets in tournaments. Posed by Geňa Hahn.

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Call a set S of vertices in an edge-coloured tournament *monochromatically absorbing* if, from any $u \in V(D) \setminus S$ there is a monochromatic directed path from u to S.

It follows from a theorem of Sands, Sauer and Woodrow [1] that any tournament not containing a ray (directed infinite path) whose arcs are coloured in two colours contains a vertex v such that $\{v\}$ is monochromatically absorbing.

Problems.

(i) Is there a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that in any tournament without a ray, edge-coloured in k colours, there is a monochromatically absorbing set of size at most $f(k)$? (It is known that if such a function exists, then $f(3) \geq 3$.)

(ii) Shen has shown that in a 3-arc-coloured finite tournament without 3-coloured triples (that is, such that between any three points two of the three arcs have the same colour) always contains a monochromatically absorbing vertex; this can be generalized to k colours. Is this true if the 3-coloured tournament has no 3-coloured directed cycles (as Shen [2] conjectures)?

(iii) If an infinite tournament without rays is arc-coloured with k colours, is there always a *finite* monochromatically absorbing set?

[1] B. Sands, N. Sauer and R. E. Woodrow, On monochromatic paths in edge-coloured digraphs. *J. Combinatorial Theory* (B) **33** (1982), 271–275.

[2] M. Shen, On monochromatic paths in m-coloured tournaments. *J. Combinatorial Theory* (B) **45** (1988), 108–111.

Problem 8 (BCC16.8): The ultimate independence ratio of a wheel. Posed by Geňa Hahn.

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Define the *independence ratio* of a graph G by $i(G) = \alpha(G)/|G|$, and the *ultimate independence ratio* by $I(G) = \lim_{k\to\infty} i(G^k)$, with $G^1 = G$ and $G^k = G \,\beta\, G^{k-1}$.

Conjecture. $I(W_5) = I(W_{2n+1}) = \frac{1}{4}$ (where W_{2n+1} is the odd wheel).

Problem 9 (BCC16.9): Colorings with minimum sum. Posed by H. Hajiabolhassan, M.L. Mehrabadi, and R. Tusserkani.

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Let G be a graph. A minimal coloring of G is a coloring which has the smallest possible sum among all proper colorings of G, using natural numbers. The *vertex–strength* of G, denoted by $s(G)$, is the minimum number of colors which is necessary to obtain a minimal coloring. Prove or disprove:

$$s(G) \leq \left\lceil \frac{\chi(G) + \Delta(G)}{2} \right\rceil .$$

We have already proved that:

$$s(G) \leq \lceil \frac{col(G) + \Delta(G)}{2} \rceil .$$

where $col(G)$ is the smallest number d such that for some linear ordering $<$ of the vertex set, the back degree $|\{v : v < u,\ vu \in E(G)\}|$ of every vertex u is strictly less than d. Therefore for all graphs G with $\chi(G) = col(G)$ e.g. all trees, the conjecture is verified. Also, we have verified this conjecture for line graphs.

Problem 10 (BCC16.10): Colouring Kempe chains. Posed by F. C. Holroyd, W. S. Leng.

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Let G be a plane graph all of whose faces except the infinite face are triangles, and whose vertex set can be expressed as the disjoint union $V_0 \cup \ldots \cup V_n$, where

(i) V_0 is a single vertex and, for each $i = 1, \ldots, n$, V_i induces a cycle;

(ii) each vertex in V_i is adjacent only to vertices in V_{i-1}, V_{i+1} and two other vertices of V_i.

Can it be shown, without assuming the four-colour theorem, that there is a proper 4-colouring of G such that each V_i receives at most three colours?

Problem 11 (BCC16.11): Linkages and flows. Posed by Bruce Reed.

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Find a short proof of the following theorem of Robertson and Seymour:

There is a function f such that, if the vertex set of a graph can be partitioned into a unique k-linkage (that is, k disjoint rooted paths as on page 101 of the proposer's paper in the Proceedings), then it can be partitioned into a unique $f(k)$-flow.

Problem 12 (BCC16.12): Second neighbourhoods in digraphs. Posed by Paul Seymour.

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Does every digraph D have a vertex v such that

$$|N^{+}(v)| \leq |N^{++}(v)|,$$

where $N^{+}(v)$ is the (out-)neighbourhood of v and $N^{++}(v)$ the strict second neighbourhood, the set of vertices reachable by directed paths of length 2 but not by single arcs from v?

The truth of this would imply that of the Caccetta–Häggkvist conjecture, according to which a digraph on n vertices with in- and out-degrees at least $n/3$ contains a directed triangle.

Remark. Gregory Gutin has pointed out that a special case of this problem, known as Dean's Conjecture, we recently solved by David C. Fisher [1].

[1] David C. Fisher, Squaring a tournament: a proof of Dean's Conjecture, *J. Graph Theory* **23** (1996), 43-48.

Problem 13 (BCC16.13): Book numbers of graphs. Posed by E. Győri.

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The *book* B_k is the graph consisting of k triangles sharing an edge. For a graph G, define $bn(G) = \max\{k : B_k \subseteq G\}$.

For $\frac{1}{2} < c < 1$, define

$$b(c) = \lim_{n\to\infty} \frac{1}{n} \min\{bn(G) : |V(G)| = n,\, d(x) \geq cn \forall x\}.$$

The problem is to determine $b(c)$. The intriguing conjecture about its value follows.

Let x be rational with $\frac{1}{2} < x < 1$. The "greedy representation" of x is given by

$$x = \frac{k_1 - 1}{k_2} \cdot \frac{k_2 - 1}{k_2} \cdots \frac{k_r - 1}{k_r},$$

where $k_i > (k_{i-1} - 1)^2$ for $i = 2, 3, \ldots, r$. (This representation is unique.) Then set

$$f(x) = \frac{k_1 - 2}{k_2} \cdot \frac{k_2 - 2}{k_2} \cdots \frac{k_r - 2}{k_r}.$$

Then f extends to a function on the real interval $(\frac{1}{2}, 1)$, continuous on every irrational. It is monotonic increasing left continuous but jumps at each rational.

Conjecture. $b(c) = f(c)$.

Remark. This is easy for $r = 1$, and is true (with a 20-page proof, see [1]) for $r = 2$.

[1] P. Erdős, R. Faudree and E. Győri, On the book size of graphs with large minimum degree, *Studia Sci. Math. Hungar.* **30** (1995), 25–46.

Problem 14 (BCC16.14): Automorphisms of lexicographic squares. Posed by Gert Sabidussi.

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Let G be a graph such that $G[G] \simeq G$, where $G[G]$ is the lexicographic product of G with itself.

Conjecture. There is an automorphism ϕ of $G[G]$ and vertices u, v, w, x, y of G such that $\phi(u, x) \in \{v\} \times V(G)$, $\phi(u, y) \in \{w\} \times V(G)$, and $v \neq w$.

Problem 15 (BCC16.15): Simultaneous edge-colourings. Posed by A. D. Keedwell.

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Suppose that $x_1, \ldots, x_m, y_1, \ldots, y_m$ are positive integers such that there exists a bipartite graph with vertex degrees $x_1, \ldots, x_m$ in one bipartite block and $y_1, \ldots, y_n$ in the other. (This is equivalent to asserting that the conditions of the Gale–Ryser theorem are satisfied.) Suppose further that all the x_i and y_j are greater than 1. Show that there is a bipartite graph having these vertex degrees, which has two proper edge-colourings such that

- for any vertex, the sets of colours appearing on edges at that vertex are the same in both colourings;
- no edge receives the same colour in both colourings.

Note. M. Mahdian, E. S. Mahmoodian, A. Saberi, M. R. Salavatipour and R. Tusserkani have made a stronger conjecture, namely, that any bridgeless bipartite graph has a pair of colourings satisfying the above properties. They have shown that this conjecture is equivalent to the celebrated 'cycle double cover conjecture' of Paul Seymour.

[1] M. Mahdian, E. S. Mahmoodian, A. Saberi, M. R. Salavatipour and R. Tusserkani, On a conjecture of Keedwell and the cycle double cover conjecture, preprint.

[2] P. D. Seymour, Sums of circuits, pp. 341–355 in *Graph Theory and Related Topics* (ed. J. A. Bondy and U. S. R. Murty), Academic Press, New York, 1979.

Problem 16 (BCC16.16): Abnormal strongly regular graphs. Posed by P. J. Cameron and P. H. Fisher.

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Call a strongly regular graph Γ *abnormal* if it contains vertices x, y, z such that $x \not\sim y$, $x \not\sim z$, $y \sim z$, and $\Gamma(x) \cap \Gamma(y) = \Gamma(x) \cap \Gamma(z)$.

Problem. Does an abnormal strongly regular graph exist?

Remark. Such a graph must have $\lambda \geq \mu$, and not all the induced subgraphs on the non-neighbours of a vertex can be edge-regular. Any strongly regular graph with $\lambda > 0$ and $\mu = 1$ is abnormal; but no such graphs are known.

Problem 17 (BCC16.17): A problem on Soicher's graph. Posed by Bill Martin.

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This problem concerns a distance-regular graph Γ on 672 vertices, based on the punctured Golay code of length 22, constructed by Leonard Soicher [1] (see Problem 13 from BCC15). It turns out that Soicher's graph, as well as being P-polynomial (that is, distance-regular), is also Q-polynomial; indeed, it is one of only two "sporadic" P- and Q-polynomial graphs (that is, not having classical parameters) which are known to the proposer. (The other is the doubled Higman–Sims graph.)

Problems. (a) Find a simple description of Γ in $\mathrm{PG}(2,4)$.

(b) Is there a Q-poset for Γ? (See below.)

Let Γ be a distance-regular graph of diameter d. Let $E_0, E_1, \ldots, E_d$ be the minimal idempotents in the Bose–Mesner algebra for Γ (in some order). The poset $(\mathcal{P}, \leq)$ of height d is a *Q-poset* for Γ if

(i) the dth level $\mathcal{P}_d$ of $\mathcal{P}$ is the vertex set of Γ;

(ii) for $0 \leq i \leq d$, the incidence matrix W_i between the ith and dth levels of $\mathcal{P}$ has constant row sums;

(iii)

$$\mathrm{rowspace}(E_i) \subseteq \mathrm{rowspace}(W_i) \subseteq \bigoplus_{j=0}^{i} \mathrm{rowspace}(E_j)$$

for $0 \le i \le d$.

Remark. The standard example: the Boolean lattice of subsets of an n-set, truncated to height d, is a Q-poset for the Johnson graph $J(n,d)$. If we require equality on the right-hand side of the inequality in (iii), and if $\operatorname{rank}(W_i) = |\mathcal{P}_i|$ (both of which occur in the standard example), then the Q-poset for Soicher's graph would satisfy

$|\mathcal{P}_0| = 1$,
$|\mathcal{P}_1| = 1 + 55 = 56$,
$|\mathcal{P}_2| = 1 + 55 + 385 = 441$,
$|\mathcal{P}_3| = 1 + 55 + 385 + 231 = 672$.

Also, presumably, the poset must be constructed from the punctured Golay code (or from PG$(2,4)$, if problem (a) is solved). Since M_{22} has no permutation action on 56 points, a construction from the punctured Golay code would require some "symmetry-breaking".

[1] L. H. Soicher, Yet another distance-regular graph related to a Golay code, *Electronic J. Combinatorics* **2** (1995), #N1.

Problem 18 (BCC16.18): The Canterbury Parades. Posed by D. A. Preece.

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The organisers of the Seventeenth British Combinatorial Conference are planning a series of parades to entertain the delegates. Seventy-six trombones will lead the parades, with one hundred and ten cornets close behind. Since the mediaeval streets of Canterbury are quite narrow, the trombonists can march four abreast, and the cornettists five abreast. It is required first that any three trombonists march in the same row in exactly one parade. This of course means constructing a resolvable 3-$(76,4,1)$ design. Such designs are known, and have 925 parallel classes, which means that the daily parades will last for about two and a half years.

The marching orders for the cornettists require a resolvable 3-$(110,5,1)$ design.

Problem. Construct such a design.

Editor's Note. The required design has 981 parallel classes. So, for the parades, it would suffice to find a "partial design" in which any three trombonists march together at most once, and there are 925 parallel classes.

In general, one could ask for solutions to the diophantine equation

$$\binom{v_1-1}{t_1-1} \Big/ \binom{k_1-1}{t_1-1} = \binom{v_2-1}{t_2-1} \Big/ \binom{k_2-1}{t_2-1},$$

and then ask for a solution to the corresponding parade problem.

Problem 19 (BCC16.19): Semi-Latin squares which are partial linear spaces. Posed by L. H. Soicher, R. A. Bailey and P. J. Cameron.

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An $(n \times n)/k$ semi-Latin square is an arrangement of nk letters in an $n \times n$ square, with k letters per cell, such that each letter occurs once in each row and once in each column. It is a *Trojan* square if it can be obtained by superimposing k mutually orthogonal $n \times n$ Latin squares. It is a *partial linear space* if no two letters occur together in the same cell more than once. Trojan squares are partial linear spaces.

Bailey (1992) showed that when $k = n - 1$ then any semi-Latin square which is a partial linear space must be Trojan, and arises from an affine plane with two distinguished parallel classes of lines.

Problem. If $k = n - 2$, must any semi-Latin square which is a partial linear space be Trojan?

Remark. Recent exhaustive checking by Soicher shows that this is true when $n = 6$ (there are no SLS-PLS or Trojan squares) and when $n = 7$.

Problem 20 (BCC16.20): Blocking-set-free configurations. Posed by H. Gropp.

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Is there a 3-chromatic linear 3-regular 3-uniform hypergraph with 16 vertices and 16 hyperedges? (Equivalently, a configuration 16_3 with no blocking set.)

Remark. This is an update of problems from earlier British Combinatorial Conferences. (See [1] for the background and current state of knowledge.) In particular, at BCC14 the proposer asked whether there is a 15_3 with no blocking set. This has been resolved negatively by unpublished results of Kel'mans, Lomonosov and Kornerup.

[1] H. Gropp, Blocking set free configurations and their relations to digraphs and hypergraphs, *Discrete Math.* **165/166** (1997), 349–360.

Problem 21 (BCC16.21): A non-Desarguesian configuration. Posed by Jane W. Di Paola.

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Prove that a non-Desarguesian projective plane must contain *Martinetti's third configuration.*

Future versions of this document will hopefully include the figure. The lines of the configuration are ABF, BCD, CAE, DEI, EFG, FDH, AGJ, BHJ, CIJ and GHI.

Remark. Killgrove has shown that each of the three non-Desarguesian planes of order 9 contains this configuration.

Problem 22 (BCC16.22): A matrix problem. Posed by David Bedford and Roger Whitaker.

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Let

$$X = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ \vdots & & & \vdots \\ B_{t1} & B_{t2} & \cdots & B_{tt} \end{pmatrix},$$

where each B_{ij} is a $n \times n$ matrix. Assume that $B_{ii} = nI$ for all i, and that, for $i \neq j$, B_{ij} has the following properties:

- entries on the leading diagonal are 1 and all other entries are 0, 1 or 2;
- B_{ij} has row and column sums n;
- $B_{ij}^\top = B_{ji}$ (and hence X is symmetric);
- $B_{ij} + B_{ji} = 2J$, where J is the $n \times n$ matrix with every entry 1.

Problem. Prove that X has nullity $t - 1$. (It is easily seen that $t - 1$ is a lower bound for the nullity.)

Problem 23 (BCC16.23): Representing orthogonal matroids. Posed by P. J. Cameron.

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Let V and W be vector spaces over a field F. Suppose that $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$ satisfy

$$\sum_{i=0}^{n} v_i \otimes w_i = 0. \tag{1}$$

Then the matroids M and M' on the ground set $\{1, \ldots, n\}$ represented by $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ are *orthogonal*, in the sense that any base of M is disjoint from some base of M' and *vice versa*.

Problem. Let M and M' be matroids on $\{1, \ldots, n\}$ which are both representable over a field F and are orthogonal in the above sense. Do there exist representations of M and M' (by $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$) over F such that (1) holds?

Problem 24 (BCC16.24): Subsums of signed permutations. Posed by J. Schönheim.

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Given a sequence $a_1, a_2, \ldots, a_n$, being a permutation of the first n positive integers, we say it has the *NZS property* if it is possible to assign $\pm$ to its members so that no subsequence of consecutive elements has a zero sum modulo $2n+1$.

(a) Which permutations of $\{1, 2, \ldots, n\}$ have the NZS property?

(b) Show that for every n there exists a latin square of order n with entries from $\{1, 2, \ldots, n\}$ such that each of its rows has the NZS property.

(c) Find for every n a sequence $a_1, a_2, \ldots, a_n$, being a permutation of the first n positive integers, such that every cyclic permutation of $a_1, a_2, \ldots, a_n$ has the NZS property.

Problem 25 (BCC16.25): Erdős–Ko–Rado at the court of King Arthur. Posed by Fred Holroyd.

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King Arthur has n knights, who have permanent places round the Table. They are to be arranged into sorties, of k knights each, such that:

(a) any two sorties intersect;

(b) because of courtly rivalries, any two knights in any sortie must sit at least d places apart round the Table (where $d \geq 2$; that is, $d = 2$ means adjacent knights cannot be in the same sortie).

Is it true that the Erdős–Ko–Rado theorem still holds, in the sense that to maximize the number of sorties, King Arthur should belong to all of them?

Remark. This is known to be true if n is sufficiently large (in terms of d and k) or if $kd \leq n < (k+1)d$.

Problem 26 (BCC16.26): The thrackle problem. Posed by J. H. Conway.

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A *thrackle* consists of a set of points in the plane called *spots*, and a set of differentiable simple curves called *paths*, such that

- the ends of each path are distinct spots, and it contains no other spot;
- any two paths must *hit* once and only once, that is, have just one common point, at which they must have different tangents;
- every spot is a hit.

Problem. Is there a thrackle with more paths than spots?

The proposer has offered \$1000 for this problem. For registered delegates of BCC16 only, the value of the prize is increased to £1000.