2.1 Prove that the ordered sum and lexicographic product of totally ordered (resp., well-ordered) sets is totally ordered (resp., well-ordered).

This involves checking the axioms, case-by-case. For the ordinal sum, we simplify the notation by using $X$ and $Y$ in place of $X \times \{0\}$ and $Y \times \{1\}$, assuming that $X$ and $Y$ are disjoint.

(a) Call the three clauses of the definition (1), (2), (3).

Irreflexivity: $z < z$ cannot be as a result of (3); if $z \in X$ then $z \not< z$ since $X$ is ordered; and if $z \in Y$ then $z \not< z$ since $Y$ is ordered.

Trichotomy: Suppose that $z_1 \neq z_2$. If $z_1, z_2 \in X$, then one of $z_1 < z_2$ and $z_2 < z_1$ holds since $X$ is totally ordered. Similarly if $z_1, z_2 \in Y$. If, say, $z_1 \in X$ and $z_2 \in Y$, then $z_1 < z_2$ by (3).

Transitivity: Suppose that $z_1 < z_2$ and $z_2 < z_3$. If $z_1, z_2, z_3 \in X$, then $z_1 < z_3$ since $X$ is ordered. So assume that at least one of the points is in $Y$. Similarly, we can assume that at least one is in $X$. Without loss of generality, $z_2 \in X$. Then $z_1 \in X$ and $z_3 \in Y$, so $z_1 < z_3$.

(b) Call the two clauses (1) and (2).

Irreflexivity: Clear.

Trichotomy: Suppose that $z_1 = (x_1, y_1) \neq z_2 = (x_2, y_2)$. If $y_1 \neq y_2$, then without loss $y_1 < y_2$, so $z_1 < z_2$ by (1). If $y_1 = y_2$, then $x_1 \neq x_2$ (property of ordered pairs); without loss, $x_1 < x_2$, and so $z_1 < z_2$ by (2).

Transitivity: Suppose that $z_1 < z_2$ and $z_2 < z_3$, where $z_i = (x_i, y_i)$. If $y_1, y_2, y_3$ are not all equal then (by considering four sub-cases) $y_1 < y_3$, so $z_1 < z_3$ by (1). Otherwise, the ordering of the $z_i$ is the same as that of the $x_i$ by (2), and transitivity for $X$ implies the result.

Now suppose that $X$ and $Y$ are well-ordered.

(a) Let $S \subseteq X \cup Y$, $S \neq \emptyset$. If $S \cap X \neq \emptyset$ then, since $X$ is well-ordered, there is a least element $s$ of $S \cap X$. By (1), $s < y$ for all $y \in S \cap Y$; so $s$ is the least element of $S$. On the other hand, if $S \cap X = \emptyset$ then $S \subseteq Y$, and so $S$ has a least element since $Y$ is well-ordered.

(b) Let $S \subseteq X \times Y$, $S \neq \emptyset$. Let

$$U = \{ y \in Y : (\exists x \in X) \text{ with } (x, y) \in S \}.$$

Then $U \neq \emptyset$, so $U$ has a least element $u$. Now let

$$T = \{ x \in X : (x, u) \in S \}.$$

Then $T$ has a least element $t$. We claim that $(t, u)$ is the least element of $S$. If $(x, y) \in S$, $(x, y) \neq (t, u)$, then either $y \neq u$ (whence $u < y$, and $(t, u) < (x, y)$ by (1)), or $y = u, x \neq t$ (whence $t < x$, and $(t, u) < (x, y) by (2)$).
2.2 Let $X$ be any set, and define $X^*$ to be the set of all finite sequences of elements of $X$. Prove that, if $X$ can be well-ordered, then so can $X^*$. Show that dictionary order on the set $X^*$ is never a well-ordering if $|X| > 1$.

If $X$ is well-ordered, then $X^2$ is well-ordered: take it to be the lexicographic product of the ordered set $X$ with itself. By induction, $X^n$ is well-ordered for all $n \geq 1$. Now $X^0$ has just one element, namely the empty sequence. Now take the ordered sum of the well-ordered sets $X^n$ for all $n$; that is, if $s \in X^n$ and $t \in X^m$, put $s < t$ if either $n < m$, or $n = m$ and $s < t$ as element of $X^n$.

Suppose that $a, b \in X$ with $a < b$. Then, in the dictionary order on $X^*$, we have the infinite decreasing sequence
\[ b > ab > aaab > aaaaab > \cdots \]

2.3 According to our definition, any natural number can be described in symbols as a sequence whose terms are the empty set $\emptyset$, opening and closing curly brackets $\{ \}$, and commas $.$. For example, the number 4 is
\[ \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \]
with eight occurrences of $\emptyset$, eight of each sort of bracket, and seven commas. How many occurrences of each symbol are there in the expression for the number $n$?

For $n \geq 1$, if $\{X\}$ is the sequence of symbols representing $n$, then $n + 1$ is represented by $\{X, \{X\}\}$. So, if $a_n, b_n, c_n, d_n$ are the numbers of empty set symbols, left braces, right braces, and commas respectively, then
\[ a_{n+1} = 2a_n, \quad b_{n+1} = 2b_n, \quad c_{n+1} = 2c_n, \quad d_{n+1} = 2d_n + 1, \]
with initial conditions
\[ a_1 = 1, \quad b_1 = 1, \quad c_1 = 1, \quad d_1 = 0. \]

By induction, the solutions are
\[ a_n = 2^{n-1}, \quad b_n = 2^{n-1}, \quad c_n = 2^{n-1}, \quad d_n = 2^{n-1} - 1, \]
for $n \geq 1$. Of course, for $n = 0$ we have $a_0 = 1$ and $b_0 = c_0 = d_0 = 0$.

2.4 Prove the properties of addition and multiplication of natural numbers used in Section 1.8.

We have to prove the following, for all natural numbers $a, b, c$:

(a) $a + b = b + a$;

(b) $a + (b + c) = (a + b) + c$;

(c) $a + 0 = a$;

(d) $a + c = b + c$ implies $a = b$;
(e) $a < b$ implies $a + c < b + c$;

(f) $ab = ba$;

(g) $a(bc) = (ab)c$;

(h) $a1 = a$;

(i) $ac = bc$ and $c \neq 0$ imply $a = b$;

(j) $ac < bc$ and $c \neq 0$ imply $a < b$.

(a) The proof is by induction on $b$. (This is induction on the well-ordered set $\omega$, that is, ordinary ‘mathematical induction’.) Both the base case and the inductive step require induction on $a$. This double induction takes great care!

Base case: we have to show that $a + 0 = 0 + a$. Since $a + 0 = a$ by definition, we must show that $0 + a = a$. This is true for $a = 0$. Suppose that $0 + a = a$. Then $0 + s(a) = s(0 + a) = s(a)$. So the statement is true, by induction on $a$.

Inductive step: we have to show that if $a + b = b + a$ for some fixed $b$ then $a + s(b) = s(b) + a$. Again this is proved by induction on $a$. Clearly it holds for $a = 0$, as in the previous paragraph. So suppose that $a + s(b) = s(b) + a$. Then

$$s(a) + s(b) = s(s(a) + b) = s(a + s(b)) = s(a + s(b)) = s(s(b) + a) = s(b) + s(a)$$

(some steps have been omitted!)

So the statement is proved.

(b) Proof by induction on $c$. For $c = 0$, we have

$$(a + b) + 0 = a + b = a + (b + 0).$$

So assume the result for $c$. Then

$$(a + b) + s(c) = s((a + b) + c) = s(a + (b + c)) = a + s(b + c) = a + (b + s(c)).$$

The result is proved.

(c) This is true by definition.

(d) First a lemma: if $s(a) = s(b)$, then $a = b$. For suppose that $s(a) = s(b)$, that is, $a \cup \{a\} = b \cup \{b\}$. If $a \neq b$, then $a \in b$ and $b \in a$, which is impossible. So $a = b$.

Induction on $c$. If $a + 0 = b + 0$, then obviously $a = b$, so the induction starts. Now suppose that it is true for $c$, and suppose that $a + s(c) = b + s(c)$. Then $s(a + c) = s(b + c)$. By our lemma, $a + c = b + c$. By the inductive hypothesis, $a = b$.

(e) Again the proof is by induction on $c$. The result is trivial for $c = 0$.

This time the required lemma is: if $s(a) < s(b)$ then $a < b$. Now $s(a) < s(b)$ means $a \cup \{a\} \subset b \cup \{b\}$, so that $a \in b$ or $a = b$. The first is impossible (since then $s(a) = s(b)$), so $a \in b$, which means $a < b$ as required.

Now suppose that $a + s(c), b + s(c)$, that is, $s(a + c) < s(b + c)$. By the lemma, $a + c < b + c$ by the inductive hypothesis, $a < b$ as required.

(f)–(j): These are multiplicative analogues of (a)–(e); the proofs are similar.
2.5 Prove that the two definitions of ordinal addition and multiplication agree.

For addition, we have to show that the sets $\alpha + \beta$ and $(\alpha \times \{0\}) \cup (\beta \times \{1\})$ are isomorphic. This can be shown by transfinite induction on $\beta$.

- For $\beta = 0$, the isomorphism between $\alpha \times \{0\}$ and $\alpha$ is clear: just throw away the tag!
- Let $\beta = s(\gamma)$ and assume that $\alpha + \gamma$ and $(\alpha \times \{0\}) \cup (\gamma \times \{1\})$ are isomorphic. Then the sets $\alpha + \beta$ and $(\alpha \times \{0\}) \cup (\beta \times \{1\})$ are obtained by adding a greatest element to each of them, and so are isomorphic.
- Suppose that $\beta$ is a limit ordinal, and that $\alpha + \gamma$ and $(\alpha \times \{0\}) \cup (\gamma \times \{1\})$ are isomorphic for all $\gamma < \alpha$. Then the union of these isomorphisms is the required isomorphism between $\alpha + \beta$ and $(\alpha \times \{0\}) \cup (\beta \times \{1\})$.

For multiplication, we have to show that $\alpha \cdot \beta$ and $\alpha \times \beta$ are isomorphic. Again we use induction on $\beta$.

- If $\beta = 0$, both sides are zero (the empty set).
- If $\beta = s(\gamma)$, then $\beta = \gamma \cup \{\gamma\}$. Assume that $\alpha \cdot \gamma$ is isomorphic to $\alpha \times \gamma$. Then
  $$\alpha \cdot \beta = \alpha \cdot \gamma + \alpha \cong \alpha \times \gamma \cup \alpha \times \{\gamma\} = \alpha \times \beta,$$
  since the elements of $\alpha \times \{\gamma\}$ are greater than those in $\alpha \times \gamma$.
- If $\beta$ is a limit ordinal, then take the union of the (unique) isomorphisms between $\alpha \cdot \gamma$ and $\alpha \times \gamma$ for $\gamma < \beta$.

2.6 Prove the following properties of ordinal arithmetic:

(a) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
(b) $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$.
(c) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.

(a) By induction on $\gamma$. Suppose that $\gamma = 0$. Then
  $$(\alpha + \beta) + 0 = \alpha + \beta = \alpha + (\beta + 0).$$
Suppose that $\gamma = s(\delta)$, and assume that $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$. Then
  $$\begin{align*}
  (\alpha + \beta) + s(\delta) &= s((\alpha + \beta) + \delta) \\
  &= s(\alpha + (\beta + \delta)) \\
  &= \alpha + s(\beta + \delta) \\
  &= \alpha + (\beta + s(\delta)).
  \end{align*}$$
Finally, suppose that $\gamma$ is a limit ordinal, and that $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$ for all $\delta < \gamma$. Then

$$
(\alpha + \beta) + \gamma = \bigcup_{\delta < \gamma}(\alpha + \beta) + \delta
= \bigcup_{\delta < \gamma}\alpha + (\beta + \delta)
= \alpha + \bigcup_{\delta < \gamma}(\beta + \delta)
= \alpha + (\beta + \gamma).
$$

(b) This question is incorrect — it should read

$$
\gamma \cdot (\alpha + \beta) = \gamma \cdot \alpha + \gamma \cdot \beta.
$$

This can be proved by induction on $\beta$, or by using the result of Exercise 2.5, as follows.

$$
\gamma \cdot (\alpha + \beta) \cong \gamma \times (\alpha + \beta)
= \gamma \times ((\alpha \times \{0\}) \cup (\beta \times \{1\}))
\cong (\gamma \times \alpha \times \{0\}) \cup (\gamma \times \beta \times \{1\})
\cong (\gamma \times \alpha) + (\gamma \times \beta).
$$

(You should check carefully that, at each stage, the obvious bijection is an order-isomorphism.) So the ordinals $\gamma \cdot (\alpha + \beta)$ and $(\gamma \times \alpha) + (\gamma \times \beta)$ are isomorphic.

For a counterexample to the version stated, note that

$$(\omega + 1) \cdot 2 = (\omega + 1) + (\omega + 1) = \omega \cdot 2 + 1$$

(since $1 + \omega = \omega$, not $\omega \cdot 2 + 2$.

(c) Proof by induction on $\gamma$:

- The result is clear if $\gamma = 0$, since $\alpha^0 = 1$.
- Suppose that $\gamma = s(\delta)$. Then

$$
\alpha^\beta \cdot s(\delta) = \alpha^{s(\beta + \delta)}
= \alpha^{\beta + \delta} \cdot \alpha
= \alpha^\beta \cdot \alpha^\delta \cdot \alpha
= \alpha^\beta \cdot \alpha^{s(\delta)}.
$$

- If $\gamma$ is a limit ordinal, take the union.

2.7 (a) Show that, if $\gamma + \alpha = \gamma + \beta$, then $\alpha = \beta$.
(b) Show that, if $\gamma \cdot \alpha = \gamma \cdot \beta$ and $\gamma \neq 0$, then $\alpha = \beta$.

(a) The identity map from $\gamma + \alpha$ to $\gamma + \beta$ maps $\gamma$ to $\gamma$ and induces an isomorphism from $\alpha$ to $\beta$. Now isomorphic ordinals are equal, by Theorem 2.3.

(b) Suppose that $\alpha < \beta$; say $\beta = \alpha + \delta$ for some $\delta > 0$. Then $\gamma \cdot \beta = \gamma \cdot \alpha + \gamma \cdot \delta$. Now it cannot be the case that $\gamma \cdot \beta = \gamma \cdot \alpha$; for the isomorphism would map $\gamma \cdot \alpha$ to a proper section of itself. Similarly, $\beta < \alpha$ is impossible. So $\alpha = \beta$. 

5
2.8 Let \((X_i)_{i \in I}\) be a family of non-empty sets. Prove that, under either of the following conditions, the cartesian product \(\prod_{i \in I} X_i\) is non-empty:

(a) \(X_i = X\) for all \(i \in I\);
(b) \(X_i\) is well-ordered for all \(i \in I\).

(a) For each \(x \in X\), the function \(f\) given by \(f(i) = x\) for all \(i \in I\) is a choice function. This shows that the cartesian product is at least as large as \(X\).

(b) Let \(x_i\) be the least element of \(X_i\). Then the function \(f\) given by \(f(i) = x_i\) for all \(i \in I\) is a choice function.

2.9 Let \(X\) be a subset of the set of real numbers, which is well-ordered by the natural order on \(\mathbb{R}\). Prove that \(X\) is finite or countable.

The well-ordered set \(X\) is isomorphic to a unique ordinal \(\alpha\); that is, \(X = \{x_\beta : \beta < \alpha\}\), and \(\beta < \gamma\) implies \(x_\beta < x_\gamma\). Choose a real number \(q_\beta\) in the interval \((x_\beta, x_{s(\beta)})\) for all \(\beta < \alpha\). (The apparent use of the Axiom of Choice here can be avoided: enumerate the rational numbers, and take the rational number with smallest index in this interval.)

These rational numbers are all distinct. For if \(\beta < \gamma < \alpha\), then \(q_\beta < x_\beta(\beta) \leq x_\gamma < q_\gamma\).

So the cardinality of \(X\) does not exceed that of \(\mathbb{Q}\).

2.10 (a) Show that any infinite ordinal can be written in the form \(\lambda + n\), where \(\lambda\) is a limit ordinal and \(n\) a natural number.

(b) Show that any limit ordinal can be written in the form \(\omega \cdot \alpha\) for some ordinal \(\alpha\).

(a) The proof is by induction. The conclusion is clear for a limit ordinal, so suppose that \(\alpha\) is a successor ordinal, say \(\alpha = s(\beta)\). By the inductive hypothesis, \(\beta = \lambda + m\), where \(\lambda\) is a limit ordinal and \(m\) a natural number. Now

\[\alpha = \beta + 1 = (\lambda + m) + 1 = \lambda + (m + 1),\]

which is of the required form.

(b) Let \(\lambda\) be a limit ordinal. By induction and part (a), every ordinal smaller than \(\lambda\) can be written in the form \(\omega \cdot \beta + n\) for some ordinal \(\beta\) and natural number \(n\). Let \(\alpha\) be the set of all the ordinals \(\beta\) which occur in such expressions. Then we have \(\beta < \alpha\), so \(\omega \cdot \beta + n < \omega \cdot \alpha\); thus, \(\lambda \leq \omega \cdot \alpha\). On the other hand, every ordinal less than \(\omega \cdot \alpha\) has the form \(\omega \cdot \beta + n\) for some \(\beta < \alpha\); so \(\omega \cdot \alpha \leq \lambda\), and we have equality.

2.11 Show that the set \(\{m - \frac{1}{n} : m, n \in \mathbb{N}, m \geq 1, n \geq 2\}\) of rational numbers is isomorphic to \(\omega^2\). Find a set of rational numbers isomorphic to \(\omega^3\).

The ordinals less than \(\omega^2\) are those of the form \(\omega \cdot m + n\). We have \(\omega \cdot m + n < \omega \cdot m' + n'\) if and only if either \(m < m'\), or \(m = m'\) and \(n < n'\). Now it is clear that the function mapping \(\omega \cdot m + n\) to \((m + 1) - \frac{1}{n+2}\) is an order-isomorphism between \(\omega^2\) and
the given set. This amounts to showing that \((m + 1) - \frac{1}{n+2} < (m' + 1) - \frac{1}{n'+2}\) if and only if either \(m < m'\), or \(m = m'\) and \(n < n'\).

To construct a set order-isomorphic to \(\omega^3\), we have to replace each interval in the above construction with a set of order-type \(\omega\). Now the interval from \((m + 1) - \frac{1}{n+2}\) to \((m + 1) - \frac{1}{n'+3}\) has length \(1/(n + 2)(n + 3)\); so take the set

\[\left\{ \left( m + 1 - \frac{1}{n + 2} - \frac{1}{(n + 2)(n + 3)(p + 2)} \right) : m, n, p \in \omega \right\} .\]

Clearly this can be extended to construct \(\omega^k\) for any \(k \in \omega\).

**2.12** Show that there are uncountably many non-isomorphic countable ordinals. Using the fact that every countable totally ordered set is isomorphic to a subset of \(\mathbb{Q}\) (see Exercise 1.16), give another proof of Cantor’s Theorem that the power set of a countable set is uncountable.

The set of countable ordinals is an ordinal, since every section of it is a countable ordinal. It cannot be a countable ordinal, else it would be smaller than itself. So it is uncountable.

By Exercise 1.17, every countable ordered set (and in particular every countable ordinal) is isomorphic to a subset of the ordered set \(\mathbb{Q}\). So \(\mathbb{Q}\) has uncountably many non-isomorphic subsets.